IV. "On a Spherical Vortex." By M. J. M. Hill, M.A., D.Sc., Professor of Mathematics at University College, London. Communicated by Professor Henrici. Received January 19, 1894.

(Abstract.)

1. In a paper published by the author in the 'Philosophical Transactions' for 1884, on the "Motion of Fluid," part of which is moving rotationally and part irrotationally, a certain case of motion symmetrical with regard to an axis was noticed (see pp. 403—405).

Taking the axis of symmetry as axis of z, and the distance of any point from it as r, and allowing for a difference of notation, it was shown that the surfaces

$$r^2\left(\frac{r^2}{a^2}+\frac{(z-Z)^2}{c^2}-1\right) = \text{constant},$$

where a, c are fixed constants and Z any arbitrary function of the time, always contain the same particles of fluid in a possible case of motion.

The surfaces are of invariable form. If the constant be negative, they are ring-shaped; if the constant be zero, the single surface represented breaks up into an evanescent cylinder and an ellipsoid of revolution; if the constant be positive, the surfaces have the axis of revolution for an asymptote.

The velocity perpendicular to the axis of symmetry is

$$2\frac{k}{c^2}r(z-Z);$$

the velocity parallel to the axis of symmetry is

$$\dot{Z}-\frac{2k}{a^2}(2r^2-a^2)-2\frac{k}{c^2}(z-Z)^2,$$

where k is a fixed constant and $\dot{Z} = dZ/dt$.

These expressions (which make the velocity infinitely great at infinity) cannot apply to a possible case of fluid motion extending to infinity. Hence the fluid moving in the above manner must be limited by a surface of finite dimensions. This limiting surface must always contain the same particles of fluid.

Where, as in the present case, the surfaces containing the same particles of fluid are of invariable form, it is possible to imagine the fluid limited by any of them, provided a rigid, frictionless boundary, having the shape of the limiting surface, be supplied and the boundary

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be supposed to move parallel to the axis of z with velocity $\dot{Z}$. Then the above expressions give the velocity components of a possible rotational motion inside the boundary. So much was pointed out in the paper cited above.

2. But a case of much greater interest is obtained when it is possible to limit the fluid moving in the above manner by one of the surfaces containing always the same particles of fluid, and to discover either an irrotational or rotational motion filling all space external to the limiting surface which is continuous with the motion inside it as regards velocity normal to the limiting surface and pressure.

3. It is the object of this paper to discuss such a case, the motion found external to the limiting surface being an irrotational motion, and the tangential velocity at the limiting surface as well as the normal velocity and the pressure being continuous.

The particular surface containing the same particles which is selected is obtained by supposing that the constant vanishes and also that $c = a$. Then this surface breaks up into the evanescent cylinder

$$r^2 = 0,$$

and the sphere $$r^2+(z-Z)^2 = a^2.$$

The molecular rotation is given by $\omega = 5kr/a^2$, so that the molecular rotation along the axis vanishes, and therefore the vortex sphere still possesses in a small degree the character of a vortex ring.

The irrotational motion outside a sphere moving in a straight line is known, and it is shown in this paper that it will be continuous with the rotational motion inside the sphere provided a certain relation be satisfied.

This relation may be expressed thus:—

The cyclic constant of the spherical vortex is five times the product of the radius of the sphere and the uniform velocity with which the vortex sphere moves along its axis.

The analytic expression of the same relation is

$$4k = 3\dot{Z}.$$

This makes $$\omega = 15\dot{Z}r/(4a^2).$$

All the particulars of the motion are placed together in the table on p. 221, in which the notation employed is as follows:—

If the velocity parallel to the axis of r be τ, and the velocity parallel to the axis of z be w, then the molecular rotation is given by

$$2\omega = \frac{d\tau}{dz} - \frac{dw}{dr}.$$

The Motion of the Spherical Vortex $r^2+(z-Z)^2=a^2$ *in an infinite Mass of Fluid parallel to the axis of* z *with uniform velocity* $\dot{Z}$.

	Rotational motion inside sphere.	At the surface of the sphere.	Irrotational motion outside sphere.
Velocity parallel to axis of r	$3\dot{Z}r(z-Z)/(2a^2)$	$\frac{3}{2}Z\sin\theta\cos\theta$	$3a^3\dot{Z}r(z-Z)/(2R^5)$
Velocity parallel to axis of z	$\dot{Z}\{5a^2-3(z-Z)^2-6r^2\}/(2a^2)$	$\dot{Z}(1-\frac{3}{2}\sin^2\theta)$	$a^3\dot{Z}\{3(z-Z)^2-R^2\}/(2R^5)$
$p/\rho+V-\Pi/\rho$	$9\dot{Z}^2[(r^2-\frac{1}{2}a^2)^2-\{(z-Z)^2-a^2\}^2+a^4]/(8a^4)$	$\frac{9}{8}\dot{Z}^2\cos^2\theta+\frac{9}{32}\dot{Z}^2$	$\frac{1}{8}\dot{Z}^2\left[\begin{array}{l}\{5-4(a/R)^3-(a/R)^6\}\\ +3\cos^2\theta\{4(a/R)^3-(a/R)^6\}+\frac{9}{4}\end{array}\right]$
Current function	$3\dot{Z}r^2\{R^2-\frac{5}{3}a^2\}/(4a^2)$	..	$-a^3\dot{Z}r^2/(2R^3)$
Surfaces containing the same particles of fluid throughout the motion	$3\dot{Z}r^2\{R^2-a^2\}/(4a^2)=\text{constant}$	..	$\dot{Z}r^2(R^3-a^3)/(2R^3)=\text{constant}$
Velocity potential	.. ..	..	$-a^3\dot{Z}(z-Z)/(2R^3)$
Molecular rotation	$15\dot{Z}r/(4a^2)$	..	
Cyclic constant of vortex...	$5a\dot{Z}$	..	
Kinetic energy............	$23\pi\rho a^3\dot{Z}^2/21$	..	$\pi\rho a^3\dot{Z}^2/3$

Also p is the pressure, ρ the density, and V the potential of the impressed forces. The minimum value of $p/\rho+\mathrm{V}$ is Π/ρ, where Π/ρ must be determined from the initial conditions.

Further R, θ are such that

$$r = \mathrm{R}\sin\theta,$$

$$z-\mathrm{Z} = \mathrm{R}\cos\theta.$$

The whole motion depends on the following constants:—

(1) The radius of the sphere a.
(2) The uniform velocity $\dot{\mathrm{Z}}$
(3) The minimum value of $p/\rho+\mathrm{V}$, viz., Π/ρ.

4. If c be not equal to a, then the surface containing the same particles when the constant vanishes breaks up into an evanescent cylinder and an ellipsoid of revolution.

Now, the velocity potential of an ellipsoid moving parallel to an axis is known. This velocity potential with a suitable relation between k and $\dot{\mathrm{Z}}$ will make the normal velocity at the surface of the ellipsoid continuous with the normal velocity of the rotational motion inside the ellipsoid, but it does not make the pressure continuous. Hence, if a motion of fluid outside the ellipsoid exist continuous with the rotational motion inside, then the motion outside the ellipsoid must be a rotational motion.

5. It cannot be argued that the application of Helmholtz's method to determine the whole motion from the distribution of vortices inside the ellipsoid must determine an irrotational motion outside the ellipsoid continuous with the rotational motion inside, because Helmholtz's method determines the irrotational motion by means of the distribution of vortices only when that distribution is known throughout space. This is not the case of the problem under discussion. For here the rotationally moving liquid has been arbitrarily limited by rejecting all the vortices outside the ellipsoid, and it is not known beforehand that the rejection of these vortices is possible.

6. Yet on account of the interest of the problem the paper contains a calculation of the velocity components in Helmholtz's manner, supposing the only vortices to be those inside the ellipsoid; *i.e.*, starting from the values of the velocity components

$$u = \frac{2k}{c^2}x(z-\mathrm{Z}),$$

$$v = \frac{2k}{c^2}y(z-\mathrm{Z}),$$

$$w = \dot{\mathrm{Z}}-\frac{2k}{a^2}(2x^2+2y^2-a^2)-2\frac{k}{c^2}(z-\mathrm{Z})^2,$$

the components of the molecular rotation are first found, viz.,

$$\xi = -k\left(\frac{4}{a^2}+\frac{1}{c^2}\right)y,$$

$$\eta = \quad \left(\frac{4}{a^2}+\frac{1}{c^2}\right)x,$$

$$\zeta = 0.$$

Then the potentials L, M, N of distributions of matter of densities $\xi/2\pi$, $\eta/2\pi$, $\zeta/2\pi$ respectively throughout the ellipsoid are determined.

These are, outside the ellipsoid,

$$\mathrm{L} = -\tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)y\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-\mathrm{Z})^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{\frac{1}{2}}},$$

$$\mathrm{M} = \tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)x\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-\mathrm{Z})^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{\frac{1}{2}}},$$

$$\mathrm{N} = 0,$$

where ϵ is the parameter of the confocal ellipsoid through x, y, z. Then

$$\frac{\partial \mathrm{N}}{\partial y}-\frac{\partial \mathrm{M}}{\partial z} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)x\,(z-\mathrm{Z})\int_\epsilon^\infty\frac{du}{(a^2+u)^2(c^2+u)^{\frac{3}{2}}},$$

$$\frac{\partial \mathrm{L}}{\partial z}-\frac{\partial \mathrm{N}}{\partial x} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)y\,(z-\mathrm{Z})\int_\epsilon^\infty\frac{du}{(a^2+u)^2(c^2+u)^{\frac{3}{2}}},$$

$$\frac{\partial \mathrm{M}}{\partial x}-\frac{\partial \mathrm{L}}{\partial y} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)\int_\epsilon^\infty\left(1-\frac{2r^2}{a^2+u}-\frac{(z-\mathrm{Z})^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{\frac{1}{2}}}.$$

To obtain the corresponding expressions inside the ellipsoid it is necessary to replace ϵ by zero.

Outside the ellipsoid $\dfrac{\partial \mathrm{N}}{\partial y}-\dfrac{\partial \mathrm{M}}{\partial z}$, $\dfrac{\partial \mathrm{L}}{\partial z}-\dfrac{\partial \mathrm{N}}{\partial x}$, $\dfrac{\partial \mathrm{M}}{\partial x}-\dfrac{\partial \mathrm{L}}{\partial y}$ are the differential coefficients of the potential function

$$-\tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)(z-\mathrm{Z})\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-\mathrm{Z})^2}{c^2+u}\right)\frac{du}{(a^2+u)\,(c^2+u)^{\frac{3}{2}}},$$

which, with a suitable value of k, gives the potential of the irrotational motion outside the ellipsoid moving parallel to the axis of z with velocity $\dot{\mathrm{Z}}$.

But inside the ellipsoid $\dfrac{\partial \mathrm{N}}{\partial y}-\dfrac{\partial \mathrm{M}}{\partial z}$, $\dfrac{\partial \mathrm{L}}{\partial z}-\dfrac{\partial \mathrm{N}}{\partial x}$, $\dfrac{\partial \mathrm{M}}{\partial x}-\dfrac{\partial \mathrm{L}}{\partial y}$ are not respectively equal to the values of u, v, w from which the investigation commenced.

In fact

$$u = \frac{\partial P}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$$

$$v = \frac{\partial P}{\partial y} + \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x},$$

$$w = \frac{\partial P}{\partial z} + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y},$$

where P is the potential function

$$\left[\frac{k}{c^2} - \tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)\,(c^2+u)^{\frac{3}{2}}}\right]\{r^2(z-Z) - \tfrac{2}{3}(z-Z)^3\}$$

$$+\left[\dot{Z} + 2k - ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2\,(c^2+u)^{\frac{1}{2}}}\right](z-Z).$$

7. The expressions $\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$, $\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$ cannot be taken by themselves to represent the velocities inside and outside the ellipsoid, for, though they would furnish continuous values of the velocities at the surface of the ellipsoid, they would not make the pressure continuous.

V. "Researches on the Structure, Organisation, and Classification of the Fossil Reptilia. Part IX. Section 1. On the Therosuchia." By H. G. SEELEY, F.R.S. Received January 8, 1894.

(Abstract.)

This paper discusses the classification of reputed Permian and Triassic Reptilia which have been referred to the Anomodontia as Theriodonts.

Lycosaurus, as the genus placed first on the Sir R. Owen's list, is accepted as the type of the Theriodontia. The species *Lycosaurus curvimola* is regarded as the type of the genus, being the only species in which its characters are fully shown. *Galesaurus planiceps*, which was the type of the Cynodontia, is probably distinct from *Nythosaurus larvatus*, and from Sir R. Owen's second type of *Galesaurus planiceps* of 1887, which is referred to as *Thrinaxodon liorhinus*. *Ælurosaurus felinus* agrees with *Galesaurus* in having a transverse development of bones of the palate, regarded as palatine and transverse, which abut toward the inner side of the lower jaw. The palate in this genus is found to be covered with groups of small teeth with conical crowns,